airfoil. A comparison between the experimental data for $C_\ell$ with and without the grid in place reinforces the credibility of the prediction model prior to stall.

In Fig. 2, similar comparisons and observations can be made for the flow past a NASA LS(1)-0417. First, note the trend of increasing $C_\ell$ as the Reynolds number is increased.[8] Second, the experimental data for $C_\ell$ show that with the grid in place there is again a significant decrease of approximately 16%. This compares favorably to the analytical model's predictions of a net decrease of 12%. With respect to the pitching moment coefficient, the turbulence-generating grid in place results in an approximately 20% reduction in the absolute value at each angle of attack.

## Conclusions

An increase in the freestream turbulent intensity from 0.6 to 1.8% results in a 15% decrease in the lift coefficient for the NACA-0012 and for the NASA LS (1)-0417. A first-order approximation for the local pressure coefficient suggests that the reduction in the lift coefficient would compare to approximately four times the change in turbulent intensity, or 12%. Thus, the predicted results compare quite favorably with the experimental evidence.

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# Solution of Aerodynamic Integral Equations Without Matrix Inversion

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## Introduction

THE formulation of aerodynamic problems often leads to the solution of a linear integral equation of a singular type. Special care is needed to obtain a satisfactory solution, particularly when the geometry is complex. The geometry of aerodynamic shapes is usually defined by a certain number of node points. Most of the known numerical methods produce a matrix equation that can be solved using any of the standard routines such as Gaussian elimination.[1] However, the matrix involved, often called a matrix of influence coefficients, is full and can be quite large. A large matrix of influence coefficients can become ill-conditioned for some applications. Also, the solution is sensitive to the node point distribution; a poor distribution will almost always fail to give reasonable results.

In what follows, a new efficient method is presented that permits control over the accuracy of the solution and is less sensitive to the node point distribution. This method has the advantage of avoiding matrix inversion. Instead, simple scalar products are utilized. Finally, this method can be implemented easily on a microcomputer and even on a programmable pocket calculator. The approach is illustrated by calculating the pressure distribution over a Joukowsky airfoil. However, this method is quite general and can have a wide range of applications.

## Analysis

The calculation of the potential flow around an airfoil can be found by solving an integral equation of the following type[2,3]:

$$\int K(s,t)\gamma(s)\mathrm{d}s = f(t) \tag{1}$$

together with the Kutta-Joukowsky condition of zero vortex strength at the trailing edge. Making use of the continuity of the vortex density function, a smooth solution, more accurate for a higher number of node points, can be constructed. Consider a set of $m$ orthogonal functions, say

$$\{\phi_k\} \qquad k=1,m$$

By substituting each of these orthogonal functions to the vortex density function into the left-hand side of Eq. (1), a new set of functions $\{g_k\}$ is obtained, defined as

$$g_k(t) = \int K(s,t)\phi_k(s)\mathrm{d}s \tag{2}$$

The functions $f$ and $g_k$, for all $k$, can be specialized to $N$ control points around the airfoil. Thus, we construct a set of vectors

$$F_0 \text{ and } \{G_k\} \qquad k=1,m$$

The initial vector $F_0$ is an element of an Euclidian space of dimension $N$; the interest is to find the closest possible approximation of $F_0$ within the subspace of dimension $m$ spanned by the vectors $G_k$.

A sequence of vectors $F_k$ given by the following recurrence relation:

$$F_k = F_{k-1} - \lambda_k G_k \tag{3}$$

are generated in such a way as to minimize the distance $|F_k|$. The shortest distance is simply given by the orthogonal projection of the extremity of vector $F_k$ into the direction of vector $G_k$.

The scalar $\lambda_k$ is then uniquely determined as

$$\lambda_k = F_{k-1}\cdot G_k / G_k \cdot G_k \tag{4}$$

The residual vector $F_m$ obtained after exhausting the whole set of vectors $\{G_k\}$ can be used to introduce a quality factor, namely

$$q = |F_m| / |F_0| \tag{5}$$

If this quality factor is not significantly smaller than unity, the process we just described can be applied iteratively to give a better approximation for the residual vector and so on. In such a case, an overall quality factor can be easily defined as the product of individual quality factors for each iteration,

$$Q = q^1 \times q^2 \times q^3 \ldots \tag{6}$$

Received July 30, 1986; revision received Oct. 20, 1986. 

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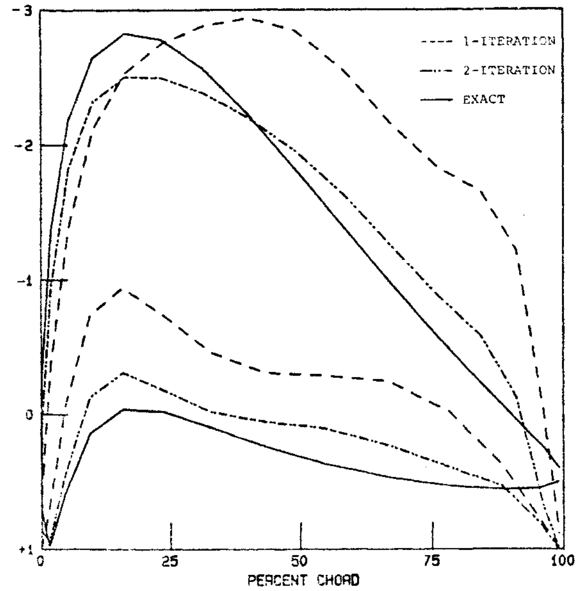


**Fig. 1 Pressure distribution of a Joukowsky airfoil at 5-deg angle of attack.**

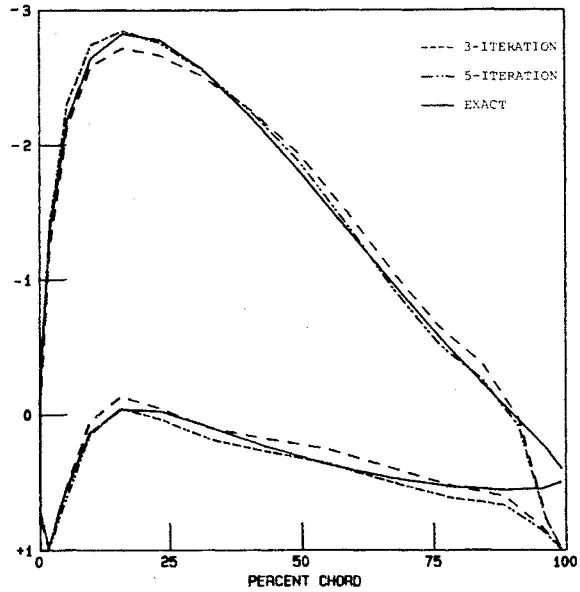


**Fig. 2 Pressure distribution of a Joukowsky airfoil at 5-deg angle of attack.**

At the same time, what we are doing is really updating the scalar coefficients as

$$\lambda_k = \lambda_k^1 + \lambda_k^2 + \lambda_k^3 + \ldots \tag{7}$$

This problem being linear, a close approximation of the vortex density function is

$$\gamma = \lambda_1 \phi_1 + \lambda_2 \phi_2 + \ldots + \lambda_m \phi_m \tag{8}$$

Linear constraint equations can be very naturally added to this method. However, a better choice is to find a set of orthogonal functions that uniquely satisfies the constraint equations when present. The number $m$ is at most equal to $N$. In practice, satisfactory solutions are computed with $m$ about half or less the number of node points $N$. It can be seen that this method has also the advantage of reducing the

**Table 1 Variation of the quality factors after each iteration**

| Quality factor | Iteration 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $q$,% | 26.6 | 32.9 | 49.1 | 71.9 | 84.8 |
| $Q$,% | 26.6 | 8.7 | 4.3 | 3.1 | 2.6 |

number of unknowns as compared to the method of solving a matrix equation.

## Results and Discussion

This method was applied to compute the pressure distribution around a nonsymmetric Joukowsky airfoil at a 5-deg angle of attack. The orthogonal functions were chosen as

$$\phi_k(s) = \sqrt{1-s^2}\,U_k(s) \qquad -1 \leq s \leq +1$$

where $U$ is a Chebyshev function of the second kind. A recurrence relation is available for this particular choice,[4] which has an obvious computational advantage in terms of speed and storage economy. Note that this particular function satisfies the Kutta-Joukowsky condition uniquely. A selection of 30 node points was taken for the geometric representation of our airfoil and 16 different functions were considered for the vortex density approximation.

Figure 1 displays the exact curve of the pressure coefficient together with the computed results of the first and second iterations at discrete points around the Joukowsky airfoil. The plotted results of the second iteration are clearly closer to the exact curve. Figure 2 shows the results for the third and fifth iteration together with the exact pressure distribution. This case demonstrates the rapid convergence toward the exact solution. The third iteration already constitutes a good approximation, but, since we have control over the accuracy, we can further improve the approximate solution by computing the fourth and fifth approximations. After the third iteration, the improvement of the solution becomes much slower, as can be seen by following the variation of the quality factors in Table 1.

From Figs. 1 and 2, it can be seen that, in the close vicinity of the trailing edge, the agreement between the exact curve and the approximate solutions is poor. This is due to the sharp zero tail angle of the Joukowsky airfoil (cusp), which does not lead the flow toward stagnation at the trailing edge. In this method, the stagnation condition is imposed at the trailing edge, which explains the discrepancy of the results in its immediate neighborhood. With a finite angle, the discrepancy disappears.

This method was implemented on an IBM-AT personal computer; the time of execution was a few seconds. Also, the method can be applied to other problems such as those involving lifting line and surface theory.

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